

# Math 2010 Week 11

## Second Derivative Test

Last time: Definiteness of sym matrices.

Thm If  $\Omega \subseteq \mathbb{R}^n$  is open,  $f: \Omega \rightarrow \mathbb{R}$  is  $C^2$ ,  
 $a \in \Omega$  is a critical point (i.e.  $\nabla f(a) = 0$ )

Then  $Hf(a)$  is

{ positive definite  $\Rightarrow a$  is a local min  
negative definite  $\Rightarrow a$  is a local max  
indefinite  $\Rightarrow a$  is a saddle point

Idea of Pf Taylor Thm,  $\nabla f(a) = 0$

$\Rightarrow$  For  $x$  near  $a$ ,

$$f(x) - f(a) \approx \frac{1}{2} (x-a)^T Hf(a) (x-a)$$

If  $Hf(a)$  is positive definite

R.H.S.  $> 0$  for all  $x \neq a$

$\Rightarrow f(x) - f(a) > 0$  for all  $x \neq a$  and near  $a$ .

$\Rightarrow f$  has a local min at  $a$ .

"Proof" is similar for the other two cases.

Geometrically

①  $Hf(a)$  is positive definite

(eg.  $f = x^2 + y^2$  at  $(0,0)$ )



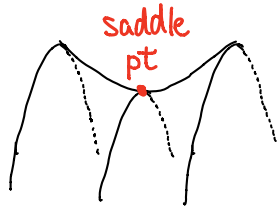
②  $Hf(a)$  is negative definite

(eg.  $f = -x^2 - y^2$  at  $(0,0)$ )



③  $Hf(a)$  is indefinite

$$\left( \begin{array}{l} \text{eg. } f = x^2 - y^2 \\ \text{at } (0,0) \end{array} \right)$$



Next: Determine definiteness of  $Hf(a)$

For the simple case  $n=2$ , it can be done easily by completing square.

Thm Let  $M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$ . Then

- $M$  is positive definite  $\Leftrightarrow AC - B^2 > 0, A > 0$
- $M$  is negative definite  $\Leftrightarrow AC - B^2 > 0, A < 0$
- $M$  is indefinite  $\Leftrightarrow AC - B^2 < 0$

Rmk  $AC - B^2 = \det M$

Pf

$$\text{Let } q(x,y) = \begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} = Ax^2 + 2Bxy + Cy^2$$

Case I ( $A \neq 0$ )

$$\begin{aligned} Aq(x,y) &= A^2x^2 + 2ABxy + ACy^2 \\ &= (Ax + By)^2 + (AC - B^2)y^2 \end{aligned}$$

Clearly

$$q(x,y) > 0 \quad \forall (x,y) \neq (0,0) \Leftrightarrow AC - B^2 > 0, A > 0$$

$$q(x,y) < 0 \quad \forall (x,y) \neq (0,0) \Leftrightarrow AC - B^2 > 0, A < 0$$

$$q(x,y) \text{ change signs} \Leftrightarrow AC - B^2 < 0$$

Case II ( $A = 0$ )  $AC - B^2 = -B^2 \leq 0$

$$q(x,y) = 2Bxy + Cy^2 = y(2Bx + Cy)$$

Clearly  $q$  is neither positive or negative definite and is indefinite  $\Leftrightarrow B \neq 0 \Leftrightarrow AC - B^2 < 0$

### Thm (Second Derivative Test)

If  $\Omega \subseteq \mathbb{R}^2$  is open,  $f: \Omega \rightarrow \mathbb{R}$  is  $C^2$ ,  $a \in \Omega$

$\nabla f(a) = 0$ . Then

- ①  $f_{xx}f_{yy} - f_{xy}^2 > 0$ ,  $f_{xx} > 0$  at  $a \Rightarrow a$  is a local min.
- ②  $f_{xx}f_{yy} - f_{xy}^2 > 0$ ,  $f_{xx} < 0$  at  $a \Rightarrow a$  is a local max.
- ③  $f_{xx}f_{yy} - f_{xy}^2 < 0$  at  $a \Rightarrow a$  is a saddle point
- ④  $f_{xx}f_{yy} - f_{xy}^2 = 0$  at  $a \Rightarrow$  inconclusive.

Rmk •  $f_{xx}f_{yy} - f_{xy}^2 = \det Hf$

- For ④,  $a$  can be local max/min or saddle pt (see eg 3)

eg 1  $f(x, y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$

Find and classify critical points of  $f$

Sol  $f$  is polynomial, so is differentiable on  $\mathbb{R}^2$

$$\nabla f = [f_x \quad f_y]$$

$$= [6x - 10y + 2 \quad -10x + 6y + 2]$$

$$\nabla f = \vec{0} \Leftrightarrow \begin{cases} 6x - 10y + 2 = 0 \\ -10x + 6y + 2 = 0 \end{cases}$$

$$\Leftrightarrow (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right)$  is the only critical point.

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ -10 & 6 \end{bmatrix}$$

$$f_{xx}f_{yy} - f_{xy}^2 = (6)^2 - (-10)^2 = -64 < 0$$

$$f_{xx} = 6 > 0$$

By 2<sup>nd</sup> derivative test,

$\left(\frac{1}{2}, \frac{1}{2}\right)$  is a saddle point

eg 2  $f(x,y) = 3x - x^3 - 3xy^2$

Find and classify critical points of  $f$

Sol  $f$  is a polynomial,  
so is differentiable on  $\mathbb{R}^2$

$$\begin{aligned}\nabla f &= [f_x \quad f_y] \\ &= [3 - 3x^2 - 3y^2 \quad -6xy]\end{aligned}$$

$$\nabla f = 0$$

$$\iff \begin{cases} 3 - 3x^2 - 3y^2 = 0 \dots \textcircled{1} \\ -6xy = 0 \dots \textcircled{2} \end{cases}$$

$$\textcircled{2} \Rightarrow x=0 \text{ or } y=0$$

$$\text{If } x=0, \textcircled{1} \Rightarrow 3 - 3y^2 = 0 \Rightarrow y = \pm 1$$

$$\text{If } y=0, \textcircled{1} \Rightarrow 3 - 3x^2 = 0 \Rightarrow x = \pm 1$$

4 critical points:  $(0, \pm 1), (\pm 1, 0)$

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$$

Critical point $a$	$Hf(a)$	$\det Hf(a)$ $f_{xx}f_{yy} - f_{xy}^2$	$f_{xx}(a)$	Nature of $a$
$(0, 1)$	$\begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$	$-36 < 0$	No need to check	Saddle point
$(0, -1)$	$\begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$	$-36 < 0$	No need to check	Saddle point
$(1, 0)$	$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$	$36 > 0$	$-6 < 0$	local max
$(-1, 0)$	$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	$36 > 0$	$6 > 0$	local min

eg 3 (Inconclusive from 2<sup>nd</sup> derivative test)

$$f(x,y) = x^2 + y^4 \quad g(x,y) = x^2 - y^4 \quad h(x,y) = -x^2 - y^4$$

$$\nabla f = [2x \ 4y^3] \quad \nabla g = [2x \ -4y^3] \quad \nabla h = [-2x \ -4y^3]$$

$\Rightarrow$   $(0,0)$  is a critical point of  $f, g, h$

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix} \quad Hg = \begin{bmatrix} 2 & 0 \\ 0 & -12y^2 \end{bmatrix} \quad Hh = \begin{bmatrix} -2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$$Hf(0,0) = Hg(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad Hh(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$  Each Hessian matrix has zero determinant at  $(0,0)$

2<sup>nd</sup> derivative test is inconclusive.

Rmk Clearly,  $f, g, h$  has local min, saddle point and local max at  $(0,0)$  respectively

## Second Derivative Test for general $n$

Let  $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  be  $C^2$ ,  $a \in \Omega$ ,  $\nabla f(a) = 0$ .

$$Hf(a) = \begin{bmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_n} \\ f_{x_2 x_1} & f_{x_2 x_2} & \cdots & f_{x_2 x_n} \\ \vdots & & \ddots & \vdots \\ f_{x_n x_1} & f_{x_n x_2} & \cdots & f_{x_n x_n} \end{bmatrix}$$

$f$  is  $C^2 \Rightarrow Hf(a)$  is symmetric. By linear algebra,

$\exists$  orthogonal  $n \times n$  matrix  $P$  (i.e.  $P^T P = I_n$ ) s.t.

$$P^T Hf(a) P = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where  $\lambda_i$  are eigenvalues of  $Hf(a)$ . Hence,

$$Hf(a) \text{ is } \begin{cases} \text{positive definite} \Leftrightarrow \text{All } \lambda_i > 0 \\ \text{negative definite} \Leftrightarrow \text{All } \lambda_i < 0 \\ \text{indefinite} \Leftrightarrow \text{Some } \lambda_i > 0, \text{ some } \lambda_j < 0 \end{cases}$$

## Another way to check definiteness of $Hf(a)$

Let  $H_k$  be the  $k$  by  $k$  submatrix

$$H_k = \begin{bmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_k} \\ f_{x_2 x_1} & f_{x_2 x_2} & \cdots & f_{x_2 x_k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_k x_1} & f_{x_k x_2} & \cdots & f_{x_k x_k} \end{bmatrix}$$

①  $Hf(a)$  is positive definite

$$\Leftrightarrow \det H_k > 0 \text{ for } k=1, 2, \dots, n$$

②  $Hf(a)$  is negative definite

$$\Leftrightarrow \det H_k \begin{cases} < 0 & \text{if } k \text{ is odd} \\ > 0 & \text{if } k \text{ is even} \end{cases}$$

For  $n=2$ ,

$$\det H_1 = \det [f_{xx}] = f_{xx}$$

$$\det H_2 = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

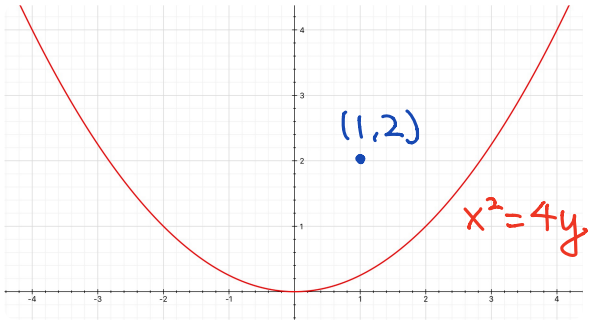
Same result as before.

# Lagrange Multipliers

Find Extrema under constraints

eg1 Find the point on the parabola

$$x^2 = 4y \text{ closest to } (1, 2)$$



Find minimum of  $f(x,y) = (x-1)^2 + (y-2)^2$

under constraint  $g(x,y) = x^2 - 4y = 0$

↑  
expressed as a level set  $g=0$

## Theorem (Lagrange Multipliers)

Let  $f, g$  be  $C^1$  functions on  $\Omega \subseteq \mathbb{R}^n$

$$S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$$

Suppose ①  $a$  is a local extremum of  $f$  on  $S$

②  $\nabla g(a) \neq 0$

Then  $\begin{cases} \nabla f(a) = \lambda \nabla g(a) \text{ for some } \lambda \in \mathbb{R} \\ g(a) = c \end{cases}$

Rmk ①  $\lambda$  is called Lagrange Multiplier

② Let  $F(x, \lambda) = f(x) - \lambda(g(x) - c)$

Then  $\nabla F(x, \lambda) = (\underbrace{\nabla f(x) - \lambda \nabla g(x)}_{n \text{ components}}, g(x) - c)$

Find critical pt of  $f$  under constraint  $g=c$   $\Leftrightarrow$  Find critical pt of  $F$  without constraint

Back to our example,

$$\text{Minimize } f(x,y) = (x-1)^2 + (y-2)^2$$

$$\text{Constraint. } g(x,y) = x^2 - 4y = 0$$

Sol

$f, g$  are differentiable on  $\mathbb{R}^2$

$$\nabla f = [2(x-1) \quad 2(y-2)]$$

$$\nabla g = [2x \quad -4] \neq \vec{0} \text{ on } \mathbb{R}^2$$

Suppose  $(x,y)$  is a local extremum of  $f(x,y)$  on  $g(x,y)=0$ .

Then by Lagrange Multipliers

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \text{ for some } \lambda \in \mathbb{R} \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 2(x-1) = 2\lambda x \quad \dots \textcircled{1} \\ 2(y-2) = -4\lambda \quad \dots \textcircled{2} \\ x^2 - 4y = 0 \quad \dots \textcircled{3} \end{cases}$$

$$\textcircled{1} \Rightarrow x-1 = \lambda x \Rightarrow x(1-\lambda) = 1$$

$$\textcircled{2} \Rightarrow y-2 = -2\lambda \Rightarrow y = 2(1-\lambda) = \frac{2}{x}$$

$$\textcircled{3} \Rightarrow x^2 - \frac{8}{x} = 0, \quad x^3 - 8 = 0 \Rightarrow x = 2$$

$$\therefore y = \frac{2}{2} = 1$$

Easy to check  $(x,y) = (2,1)$  is a solution

Geometrically,  $f$  must have a min. on  $g=0$

By Lagrange Multipliers, only one pt can be that min.

$\Rightarrow f$  has minimum at  $(2,1)$  on  $g=0$ .



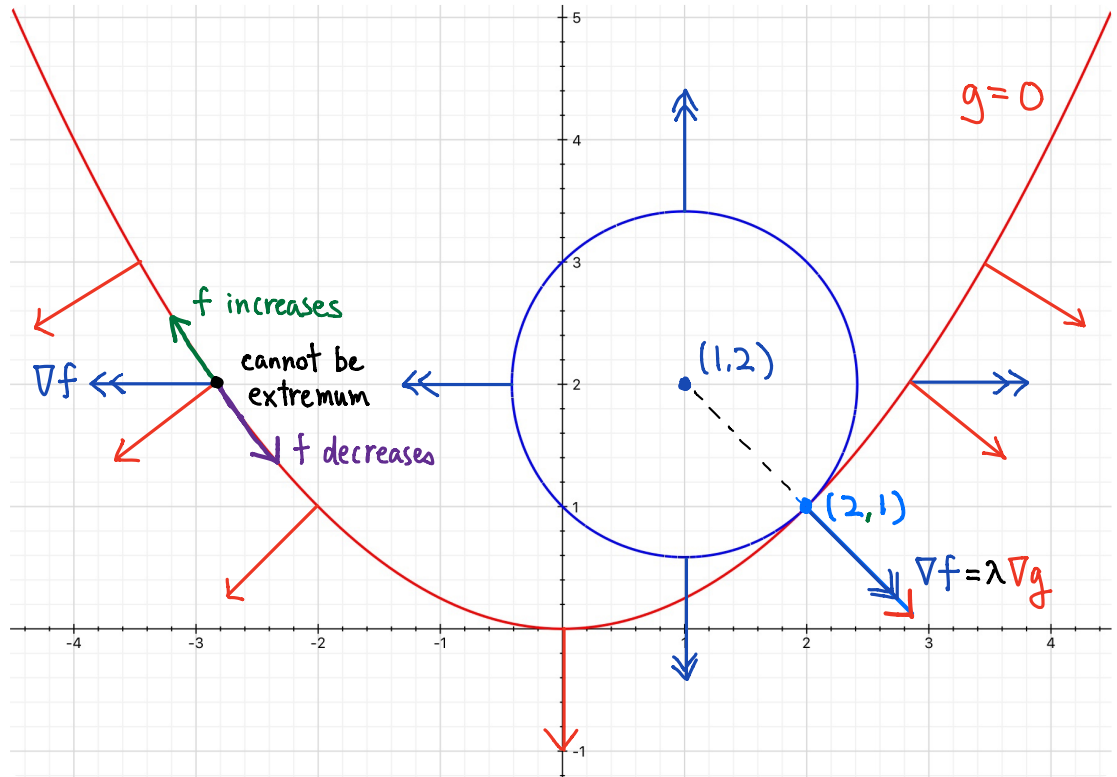
$$f(x,y) = (x-1)^2 + (y-2)^2$$

$$g(x,y) = x^2 - 4y$$

→ direction of  $\nabla g$  ( $\perp$  level set of  $g$ )

→ direction of  $\nabla f$  (direction which  $f$  increases most rapidly)

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$



Ex Find the point on the parabola  $x^2 = 4y$  closest to  $(2, 5)$

$$f(x, y) = (x-2)^2 + (y-5)^2$$

$$g(x, y) = x^2 - 4y$$

Rmk

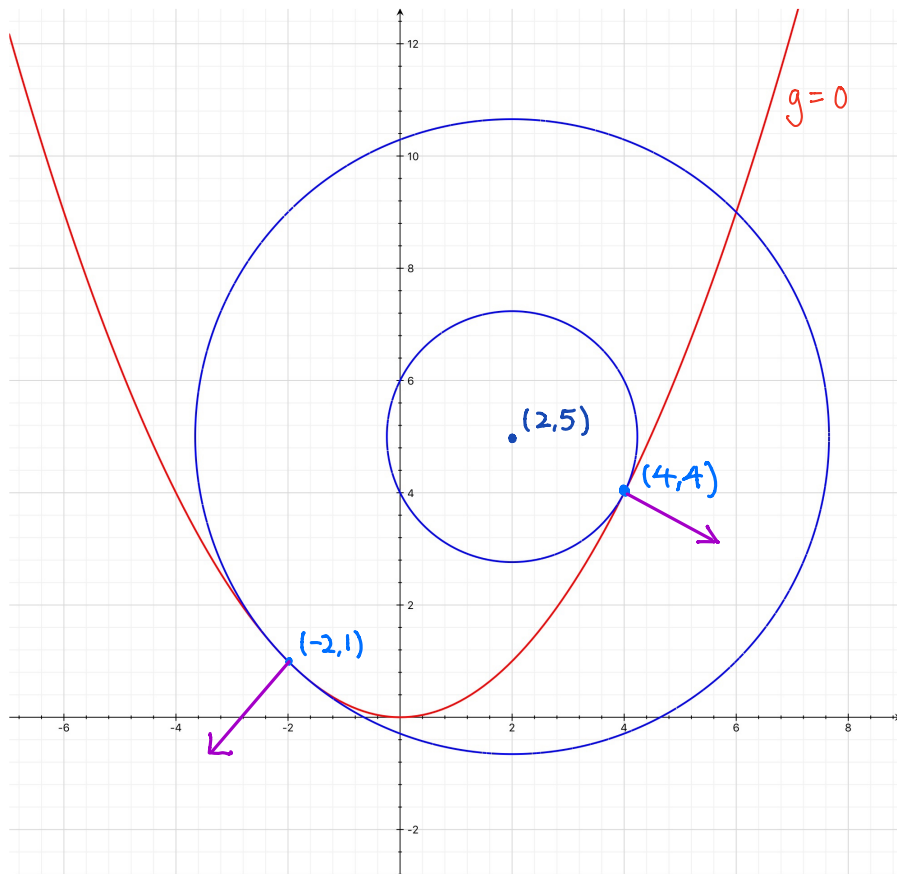
$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \text{ has solutions}$$

$$(4, 4)$$

↑  
global min  
on  $g=0$

$$(-2, 1)$$

↑  
Not local extremum  
on  $g=0$



eg<sup>2</sup> Maximize  $xy^2$  on the ellipse

$$x^2 + 4y^2 = 4$$

Sol Let  $f(x,y) = xy^2$

$$g(x,y) = x^2 + 4y^2$$

Note  $f$  is continuous and the ellipse  $g=4$  is closed and bounded

By EVT,

$f$  has global max and min. on  $g=4$

$$\nabla f = [y^2 \quad 2xy]$$

$$\nabla g = [2x \quad 8y]$$

Note  $\nabla g \neq 0$  on  $x^2 + 4y^2 = 4$

Lagrange Multipliers:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \Leftrightarrow \begin{cases} y^2 = 2\lambda x & \dots \textcircled{1} \\ 2xy = 8\lambda y & \dots \textcircled{2} \\ x^2 + 4y^2 = 4 & \dots \textcircled{3} \end{cases}$$

Case 1 If  $y=0$ , then  $\textcircled{3} \Rightarrow x^2=4$

$$\Rightarrow x = \pm 2, \lambda = 0 \text{ by } \textcircled{1}$$

$$\therefore (x,y) = (\pm 2, 0)$$

Case 2 If  $y \neq 0$ , then

$$\frac{\textcircled{2}}{\textcircled{1}} \Rightarrow \frac{2xy}{y^2} = \frac{8\lambda y}{2\lambda x} \Rightarrow \frac{2x}{y} = \frac{4y}{x} \Rightarrow x^2 = 2y^2$$

$$\text{By } \textcircled{3}, 6y^2 = 4 \Rightarrow y = \pm \sqrt{\frac{2}{3}}$$

$$\therefore x^2 = 2y^2 = \frac{4}{3} \Rightarrow x = \pm \sqrt{\frac{4}{3}}$$

$$\therefore (x,y) = \left( \pm \sqrt{\frac{4}{3}}, \pm \sqrt{\frac{2}{3}} \right)$$

Comparing values of  $f$  at the 6 points found using Lagrange Multipliers.

$$f(x, y) = x y^2$$

$$f(\pm 2, 0) = 0$$

$$f\left(\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right) = \sqrt{\frac{4}{3}} \cdot \frac{2}{3} = \frac{4}{3\sqrt{3}} \leftarrow \text{max}$$

$$f\left(-\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\sqrt{\frac{4}{3}} \cdot \frac{2}{3} = -\frac{4}{3\sqrt{3}} \leftarrow \text{min}$$

For  $f(x, y)$  on  $g=4$ ,

$$\text{Global max value} = \frac{4}{3\sqrt{3}} \text{ at } \left(\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right)$$

$$\text{Global min value} = -\frac{4}{3\sqrt{3}} \text{ at } \left(-\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right)$$

Rmk We may use another form of Lagrange Multiplier

$$\begin{aligned} \text{Let } F(x, y, \lambda) &= f(x, y) - \lambda(g(x, y) - 4) \\ &= x y^2 - \lambda(x^2 + 4y^2 - 4) \end{aligned}$$

$$\nabla F = (y^2 - 2\lambda x, 2xy - 8\lambda y, x^2 + 4y^2 - 4)$$

$$\nabla F = 0 \Leftrightarrow \begin{cases} y^2 - 2\lambda x = 0 \\ 2xy - 8\lambda y = 0 \\ x^2 + 4y^2 - 4 = 0 \end{cases}$$

Same system as before

For problems of finding max/min of  $f: A \rightarrow \mathbb{R}$ , Lagrange Multipliers can be used to study  $f$  on  $\partial A$

Redo an example before:

eg Find global max/min of

$$f(x,y) = x^2 + 2y^2 - x + 3 \text{ for } x^2 + y^2 \leq 1$$

Sol Domain =  $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

As found before,  $f$  has only one critical point  $(\frac{1}{2}, 0)$  in  $\text{int}(A)$  with

$$f(\frac{1}{2}, 0) = \frac{11}{4}$$

To study  $f$  on  $\partial A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  by Lagrange Multipliers:

$$\text{Let } g(x,y) = x^2 + y^2$$

$$\nabla g = (2x, 2y) \neq \vec{0} \text{ on } \partial A \ (g=1)$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g=1 \end{cases} \Leftrightarrow \begin{cases} 2x-1 = 2\lambda x \dots \textcircled{1} \\ 4y = 2\lambda y \dots \textcircled{2} \\ x^2 + y^2 = 1 \dots \textcircled{3} \end{cases}$$

$$\textcircled{2} \Rightarrow (4 - 2\lambda)y = 0$$

$$\Rightarrow \lambda = 2 \quad \text{or} \quad y = 0$$

$$\text{By } \textcircled{1}, 2x-1 = 4x$$

$$x = -\frac{1}{2}$$

$$\text{By } \textcircled{3}, x = \pm 1$$

$$\text{By } \textcircled{3}, y = \pm \frac{\sqrt{3}}{2}$$

Comparing the values of  $f$  at these points

$$f\left(\frac{1}{2}, 0\right) = \frac{11}{4}$$

$$f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{21}{4}$$

$$f(1, 0) = 3$$

$$f(-1, 0) = 5$$

Hence, Max value =  $\frac{21}{4}$  at  $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Min value =  $\frac{11}{4}$  at  $\left(\frac{1}{2}, 0\right)$